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# Finite Groups with Small Sums of Degrees of Some Non-linear Irreducible Characters

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We obtain lower estimates for sums of degrees of some non-linear irreducible characters of non-$p$-nilpotent groups, and investigate the structure of groups for which these estimates are attained. 

Let $p$ be a prime divisor of the order of a finite group $G$, $P$ a fixed Sylow $p$-subgroup of $G$, $\mathrm{Irr}(G)$ the set of all ordinary irreducible characters of $G$, and $\mathrm{Irr}_1(G)$ the subset of all non-linear characters of $\mathrm{Irr}(G)$. A group $G$ is said to be $p$-nilpotent if it has a normal $p$-complement. Let

$$\begin{aligned}
\mathrm{Irr}_1(G,p') &= \{\chi \in \mathrm{Irr}_1(G) \mid p \nmid \chi(1)\},\\
\mathrm{Irr}_1(G,P,p') &= \{\chi \in \mathrm{Irr}_1(G,p') \mid p \mid |G/\ker\chi|\},\\
\mathrm{Irr}_1^0(G,p') &= \{\chi \in \mathrm{Irr}_1(G,p') \mid G/\ker\chi \text{ is not } p\text{-nilpotent}\},\\
\tau_1(G,p') &= \textstyle\sum \chi, \qquad \text{where } \chi \text{ runs over the set } \mathrm{Irr}_1(G,p'),\\
T_1(G,p') &= \tau_1(G,p')(1).
\end{aligned}$$

If we take, in the last two definitions, $\mathrm{Irr}_1(G,P,p')$ or $\mathrm{Irr}_1^0(G,p')$) instead of $\mathrm{Irr}_1(G,p')$, we obtain the definitions of

$$\tau_1(G,P,p'), \qquad T_1(G,P,p')\big(\tau_1^0(G,p'), \qquad T_1^0(G,p')\big).$$

In this paper we obtain lower estimates of the numbers defined above and investigate the groups for which these estimates are attained.

Thompson (see [Is, Corollary 12.2]) showed that if $T_1(G,p') = 0$ then $G$ is $p$-nilpotent. Our results generalize Thompson's theorem. Proposition 9 contains a new property of Thompson's groups. Moreover, as follows from this proposition (see Remark 1 following it), Thompson's groups are

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solvable. The proof of this fact uses the classification of finite simple groups.

Let $G'$ be the commutator subgroup of $G$, $\Phi(G)$ the Frattini subgroup of $G$, $Z(G)$ the centre of $G$. $(A, B)$ a Frobenius group with kernel $B$ and a complement $A$, $C(m)$ a cyclic group of order $m$, and $E(p^m)$ an elementary abelian group of order $p^m$. Denote by $\mathrm{Irr}(\chi)$ the set of all irreducible constituents of a character $\chi$. Let $\mathrm{Lin}(G)$ be the set of all linear characters of $G$. In the sequel $P$ denotes a fixed Sylow $p$-subgroup of $G$. If $X$ is a set of characters of a group then $X^{\#} = X - \{\varepsilon\}$, where $\varepsilon$ is the principal character of this group. Next, $O^p(G)$ denotes the subgroup generated by all $p'$-elements of $G$.

## 1

In this section we prove some introductory lemmas.

LEMMA 1. *Suppose that $P'$ is normal in $G$. Then $G$ is $p$-nilpotent if and only if $G/P'$ is $p$-nilpotent.*

This is obvious since $P' \leq \Phi(G)$.

LEMMA 2. *Suppose that $P$ is normal in $G$. Then*

$$T_1(G, P, p') = T_1(G/P', P/P', p').$$

*Proof.* Since $\mathrm{Irr}_1(G/P', P/P', p') \subseteq \mathrm{Irr}_1(G, P, p')$, it is sufficient to prove the inverse inclusion. Take $\chi = \mathrm{Irr}_1(G, P, p')$, $\lambda \in \mathrm{Irr}(\chi_P)$. Since $p \nmid \chi(1)$ and $\lambda(1) | \chi(1)$, $p \nmid \lambda(1)$ and $\lambda \in \mathrm{Lin}(P)$, and hence $P' \leq \ker \lambda$ and $P' \leq \ker \chi$ (since $\lambda$ is an arbitrary irreducible constituent of $\chi$). Therefore $\chi$ is contained in $\mathrm{Irr}(G/P', P/P', p')$. ∎

LEMMA 3. *Suppose that $P$ is normal in a non-$p$-nilpotent group $G$. If $p^\alpha \,|\, |G : G'|$ but $p^{\alpha+1} \nmid G : G'|$, then*

(a) $T_1(G, P, p') \geq |P : P'| - p^\alpha$.

(b) $T_1(G, P, p') = |P : P'| - p^\alpha$ *if and only if*

$$G/P' = (H, P_1/P') \times P_2/P', |P_2/P'| = p^\alpha;$$

*here $P_1 \in \mathrm{Syl}_p(G')$.*

*Proof.* In view of Lemmas 1 and 2 we may assume $P' = 1$, i.e., $P$ is abelian.

(a, i) Assume that $\alpha = 0$, i.e., $P \leq G'$. Take $\lambda \in \mathrm{Lin}^{\#}(P)$. By virtue of $P \leq G'$ one has $\mathrm{Irr}(\lambda^G) \subseteq \mathrm{Irr}_1(G)$. Now $\mathrm{Irr}_1(G) = \mathrm{Irr}_1(G, p')$ by Ito's theorem on degrees [Is, Theorem 6.15]. Take $\chi \in \mathrm{Irr}(\lambda^G)$. Then $P \cap \ker \chi = \ker \chi_P \leq \ker \lambda$ by reciprocity, so $P$ is not contained in $\ker \chi$. Hence $\chi \in \mathrm{Irr}_1(G, P, p')$. By reciprocity

$$\mathrm{Lin}^{\#}(P) \subseteq \mathrm{Irr}(\tau_1(G, P, p)_P),$$
$$|P| - 1 = |\mathrm{Lin}^{\#}(P)| \leq T_1(G, P, p')|$$

and (a) is true for $\alpha = 0$.

(a, ii) Assume $\alpha > 0$. Take a $p'$-Hall subgroup $H$ of $G$ (Schur–Zassenhaus). Since $HG'$ is normal in $G$ one obtains $G = N_G(H)HG' = N_G(H)G'$ (the Frattini lemma and the Schur–Zassenhaus theorem). Now $N_G(H) = H \times P_2$, where $P_2 = P \cap N_G(H)$. In view of $C_G(P_2) \geq \langle P, H \rangle = G$ one has $P_2 \leq Z(G)$. Since $P$ is abelian, $P \cap G' \cap Z(G) = 1$ (the transfer theorem), and we obtain

$$P_2 \cap G' = 1, \qquad G = HG' \times P_2, \qquad |P_2| = p^{\alpha}.$$

Take $P_1 \in \mathrm{Syl}_p(G')$. In view of commutativity of $P$ one has $O^p(HG') = HG'$, so $P_1 \leq (HG')'$. Hence (a, i) implies $T_1(HG', P_1, p') \geq |P_1| - 1$. If $\lambda \in \mathrm{Irr}_1(HG', P_1, p')$, $\mu \in \mathrm{Lin}(P_2)$, then $\lambda \times \mu \in \mathrm{Irr}_1(G, P, p')$, implying that

$$T_1(G, P, p') \geq T_1(HG', P_1, p')|P_2| \geq (|P_1| - 1)|P_2| = |P| - p^{\alpha}$$

and (a) is proved.

(b, i) Assume that $\alpha = 0$, i.e., $P \leq G'$. By this assumption $T_1(G, P, p') = |P| - 1$. It follows from the proof of (a, i) that $\lambda^G \in \mathrm{Irr}(G)$ for any $\lambda \in \mathrm{Lin}^{\#}(P)$. So [K, Corollary 37.5.4] one has $G = (H, P)$. The inverse follows from the description of irreducible characters of Frobenius groups [Is, Theorem 6.34].

(b, ii) Assume $\alpha > 0$. As in (a, ii) one has $G = HG' \times P_2, |P_2| = p^{\alpha}$ Let $P_1 \in \mathrm{Syl}_p(G')$. It follows from (a) and $(|P_1| - 1)P_2| \leq T(HG', P, p')|P_2| \leq T(G, P, p') = |P| - p^{\alpha}$ that $T_1(HG', P_1, p') = |P_1| - 1$. Then (b, i) implies $HG' = (H, G')$. On the other hand, if $G$ has the structure described above, then $T_1(G, P, p') = |P| - p^{\alpha}$. ∎

LEMMA 4. *Suppose that* $P < H < G$. *Then*

(a) $T_1(H, P, p') \leq T_1(G, P, p')$.

(b) $T_1(H, p') \leq T_1(G, p')$.

*Proof.* (a) Take $\lambda \in \mathrm{Irr}_1(H, P, p')$. Since $p \nmid \lambda^G(1) = |G : H|\lambda(1)$ there is $\chi \in \mathrm{Irr}(\lambda^G)$ with $p \nmid \chi(1)$. By reciprocity $\chi(1) \geq \lambda(1)$, $H \cap \ker \chi \leq \ker \lambda$. Since $P$ is not contained in $\ker \lambda$ it is also not contained in $\ker \chi$ and $\chi \in \mathrm{Irr}_1(G, P, p')$. By reciprocity

$$\mathrm{Irr}_1(H, P, p') \subseteq \mathrm{Irr}(\tau_1(G, P, p')_H)$$

and the result follows.

(b) is proved similarly. ∎

LEMMA 5. *If G is not p-nilpotent then* $T_1^0(G, p') \geq p - 1$.

*Proof.* If $M$ is normal in $G$ then $T_1^0(G, p') \geq T_1^0(G/M, p')$. Hence we may assume that all proper epimorphic images of $G$ are $p$-nilpotent. Then $G$ contains only one minimal normal subgroup $M$. Since $G/M$ is $p$-nilpotent, $p \mid |M|$. Now $M \leq G'$ as the unique minimal normal subgroup of a non-abelian group $G$. Let us consider the following two cases.

(i) $p \nmid |G/M|$.

Then $P \leq M$. If $\chi \in \mathrm{Irr}_1(G, P, p')$ then $P$ and $M$ are not contained in $\ker \chi$, so that $\chi$ is faithful and $G/\ker \chi = G$ is not $p$-nilpotent. Hence $\mathrm{Irr}_1^0(G, p') = \mathrm{Irr}_1(G, P, p')$. If $\lambda \in \mathrm{Lin}^{\#}(P)$ then by virtue of $p \nmid \lambda^G(1)$ there is $\chi \in \mathrm{Irr}(\lambda^G)$ with $p \nmid \chi(1)$. Obviously

$$\chi \in \mathrm{Irr}_1(G, P, p') = \mathrm{Irr}_1^0(G, p'),$$

so, by reciprocity, one obtains

$$T_1^0(G, p') \geq |\mathrm{Lin}^{\#}(P)| = |P : P'| - 1 \geq p - 1.$$

(ii) $p \mid |G : M|$.

Let $H/M$ be a normal $p$-complement of $G/M$. Since $G/H$ is a $p$-group, $H$ is not $p$-nilpotent. By Tate's theorem [Is, Theorem 6.31] the intersection $H \cap P = P_1$ is not contained in $P'$. Hence there is $\lambda \in \mathrm{Lin}(P)$ such that $P_1$ is not contained in $\ker \lambda$. Take $\chi \in \mathrm{Irr}(\lambda^G)$. Since $P_1 \leq G'$ and $P_1$ is not contained in $\ker \chi$ by reciprocity we have $\chi(1) > 1$. Since $p \nmid \lambda^G(1)$ we may choose $\chi$ so that $p \nmid \chi(1)$. Since $P_1$ is not contained in $\ker \chi$, $M$ is not contained in $\ker \chi$, and $\chi$ is faithful. Since $G/\ker \chi = G$ is not $p$-nilpotent, $\chi \in \mathrm{Irr}_1^0(G, p')$. As $\lambda$ is an arbitrary character from $\mathrm{Lin}(P/P') - \mathrm{Lin}(P/P_1P')$, there are $|P : P'| - |P : P_1P'| \geq p - 1$ ways to choose $\lambda$. Hence, by reciprocity, $T_1^0(G, p') \geq p - 1$, and the lemma is proved. ∎

LEMMA 6. *If $P < H < G$ then*

$$T_1^0(H, p') \leq T_1^0(G, p').$$

*Proof.* Take $\lambda \in \mathrm{Irr}_1^0(H, p')$. As $p \nmid \lambda^G(1)$ there exists $\chi \in \mathrm{Irr}(\lambda^G)$ with $p \nmid \chi(1)$. Since

$$H \cap \ker \chi = \ker \chi_H \leq \ker \lambda,$$

by reciprocity the group

$$H/\ker \chi_H = H/H \cap \ker \chi \cong H \ker \chi/\ker \chi$$

is not $p$-nilpotent because some of its epimorphic image is the non-$p$-nilpotent group $H/\ker \lambda$. Then $G/\ker \chi$ is not $p$-nilpotent: it contains the non-$p$-nilpotent subgroup $H \ker \chi/\ker \chi$. Thus $\chi$ is contained in $\mathrm{Irr}_1^0(G, p')$ and the result follows by reciprocity. ∎

LEMMA 7. *If $G$ is not $p$-nilpotent then $T_1(G, P, p') \geq |P : P'| - p^\alpha$, where $p^\alpha$ is the order of a Sylow $p$-subgroup of $G/G'$.*

*Proof.* Let $H/G'$ be a normal $p$-complement of $G/G'$. Then $|G : H| = p^\alpha$ and $H$ is not $p$-nilpotent. Thus [Is, Theorem 6.31] $P_1 = H \cap P$ is not contained in $P'$, and there is $\lambda \in \mathrm{Lin}(P)$ such that $P_1$ is not contained in $\ker \lambda$. The number of such $\lambda$ is equal to

$$|P : P'| - |P : P_1 P'| = |P : P'| - p^\alpha.$$

As above, there exists $\chi \in \mathrm{Irr}(\lambda^G) \cap \mathrm{Irr}_1(G, P, p')$, and the result follows by reciprocity. ∎

*Remark.* Lemma 7 generalizes Lemma 3(a).

LEMMA 8. *If $P$ is cyclic and $G$ is not $p$-nilpotent then $P \leq G'$.*

*Proof.* This is a corollary of Tate's theorem [Is, Theorem 6.31]. However, we deduce the lemma from more elementary results. Suppose that $P$ is not contained in $G'$. We may assume that $P > P \cap G' > 1$. Because $P \cap G' \cap Z(G) = 1$ we have $P \cap Z(G) = 1$. Take in $G/G'$ a normal subgroup $H/P'$ of index $p$. Set $F = N_G(P)$ and take $T$, a $p'$-Hall subgroup of $F$ (Schur–Zassenhaus). Now $F \cap H$ is a normal subgroup of index $p$ in $F$ and $T \leq F \cap H$. By the Frattini Lemma and Schur–

Zassenhaus one obtains

$$F = (F \cap H)N_F(T) = (P \cap H)TN_F((T) = (P \cap H)N_F(T).$$

Since $N_F(T) = T \times P_1$, where $1 < P_1 \leq P, C_F(P_1) \geq \langle P, T \rangle = F$. By the Burnside normal $p$-complement theorem $F$ is not $p$-nilpotent. So $P_1 = 1$ since by the above $p \nmid |Z(F)|$, a contradiction. ∎

A group $G$ with $T_1(G, p') = 0$ deserves special investigation. Put $\mathrm{Irr}(G, p') = \mathrm{Irr}_1(G, p') \cup \mathrm{Lin}(G)$.

PROPOSITION 9. *Suppose that a group $G$ satisfies $T_1(G, p') = 0$. If $P \in \mathrm{Syl}_p(G)$, $N = N_G(P)$ then the following assertions hold*:

(a) $N \cap G' = P'$.

(b) $N = P \times A$, *where $A$ is abelian.*

(c) $G = AL$, *a semi-direct product; here $L = P^G$ is the normal closure of $P$ in $G$,*

(d) $N_L(P) = P$.

*Proof.* Take $\lambda \in \mathrm{Irr}(N, p')$. Because $p \nmid \lambda^G(1)$ there exists $\lambda^0$ in $\mathrm{Irr}(\lambda^G) \cap \mathrm{Irr}(G, p')$. By the condition $\lambda^0(1) = 1$. In particular $T_1(N, p') = 0$. Now $D = N \cap (\cap \ker \lambda^0) = \cap \ker \lambda = N'$ (here $\lambda$ runs over the set $\mathrm{Lin}(N)$). By the above $N' \geq N \cap G' \geq N'$ so $N' = N \cap G'$.

Using this reasoning we may take instead of $N$ any subgroup $H$ of $G$ such that $P \leq H$. So $P' = P \cap G'$. Because $P' \leq P \cap N' \leq P \cap G' = P'$ one has $P' = P \cap N'$. By what has been proved $N/P$ is abelian. Hence $N' \leq P$ and $P' = P \cap N' = N'$. Thus $P' = N' = N \cap G'$ and (a) is proved.

Now $P' \leq \Phi(P) \leq \Phi(N)$. Since $N/P' = N/N'$ is abelian, $N$ is nilpotent [G], $N = P \times A$. Now $A$ is abelian, and (b) is proved.

Let $L = P^G$, the normal closure of $P$ in $G$. Then $G = LN$ (Frattini). Set $M = N \cap L = N_L(P)$ and assume that $P < M$. Take $\lambda \in \mathrm{Irr}^\#(M/P) = \mathrm{Lin}^\#(M/P)$. Since $p \nmid \lambda^G(1)$ there exists $\lambda^0 \in \mathrm{Irr}(\lambda^G) \cap \mathrm{Irr}(G, p')$ and $\lambda^0(1) = 1$ by the condition. Thus $\lambda^0_M = \lambda$, $M \cap \ker \lambda^0 = \ker \lambda \geq P$; hence $L \leq \ker \lambda^0$ for any choice of $\lambda$. Since $M$ is not contained in $\ker \lambda$, $M$ is not contained in $\ker \lambda^0$. Moreover $L$ is not contained in $\ker \lambda^0$, a contradiction. Thus $M = N \cap L = P$ and (d) is proved. In particular $A \cap L = 1$. Now $G = LN = LPA = LA$ implies $G = AL = AP^G$, a semi-direct product. The proposition is proved. ∎

*Remark* 1. If a group $G = PH$ is a semi-direct product with $P \in \mathrm{Syl}_p(G)$ is such that $N_G(P) = P$ then as known $G$ is solvable (Mann and Shalev [MaS] also reported to me the proof of this result). This proof uses the classification of finite simple groups. Hence Proposition 9 shows that $G$ with $T_1(G, p') = 0$ is solvable. In fact, by Proposition 9, $P$ coincides with its normalizer in $P^G$. Since $P^G$ is $p$-nilpotent (Thompson) it is solvable. Now $G$ is solvable since $G/P^G$ is abelian (Proposition 9).

*Remark* 2. Let $G(p') = \cap \ker \chi$, where $\chi$ runs over the set $\mathrm{Irr}_1(G, p')$. It is well known [B1] that $G(p')$ is $p$-nilpotent. Now we prove that $\mathrm{Irr}_1(PG(p'), p')$ is empty for $P \in \mathrm{Syl}_p(G)$, i.e., $G(p')$ is solvable (Remark 1). Take $\lambda \in \mathrm{Irr}_1(PG(p'), p'), \chi \in \mathrm{Irr}(\lambda^G)$. Because $G(p')$ is not contained in $\ker \chi$ and $\chi$ is not linear, $p|\chi(1)$. Hence $p$ divides the degrees of all irreducible constituents of $\lambda^G$. Therefore

$$|G : PG(p')|\lambda(1) = \lambda^G(1) \equiv 0 (\mathrm{mod}\ p).$$

Since $p \nmid |G : PG(p')|$, $p|\lambda(1)$. Thus $\mathrm{Irr}_1(PG(p'), p')$ is empty and $PG(p')$ is solvable by the previous remark. In particular $G(p')$ is solvable.

*Remark* 3. Let $G(p) = \bigcap \ker \chi$, where $\chi$ runs over the set $\mathrm{Irr}_1(G(p)) = \{\tau \in \mathrm{Irr}(G) | p | \tau(1)\}$. If $\lambda \in \mathrm{Irr}_1(G(p)), \chi \in \mathrm{Irr}(\lambda^G)$ then $p \nmid \lambda(1)$ since $G(p)$ is not contained in $\ker \chi$. Therefore $p \nmid \lambda(1)$. Thus one has

$$\mathrm{Irr}_1(G(p)) = \mathrm{Irr}_1(G(p), p').$$

As Michler [Mic] showed a Sylow $p$-subgroup of $G(p)$ is abelian and normal (but it is possible for $G(p)$ to be non-solvable).

In the sequel $\varphi(*)$ denotes the Euler number theoretic function.

Corollary. *If $G$ is not $p$-nilpotent then*

$$T_1(G, P, p') \geq \varphi(|P : P'|).$$

*Proof.* This follows from Lemma 3(a). ∎

Lemma 10. *Suppose that $G$ is not $p$-nilpotent and that $P$ is abelian. If $T_1(G, P, p') = \varphi(|P : P'|)$ then the following assertions hold*:

(a) $N_{G'P}(P) = P$.

(b) $N_G(P) = (C(s), C(p)) \times P_2$, *where* $|P : P_2| = p$.

(c) $G/G' \cong C(s) \times P_2$.

*Proof.* Set $N = N_G(P)$, $N_1 = N_{G'P}(P) = N \cap G'P$. Since $G = (G'P)N = G'N, G/G' \cong N/N \cap G'$, and (c) follows from (a) and (b) in view of the transfer theorem. It remains to prove (a) and (b).

By the Burnside normal $p$-complement theorem $N$ is not $p$-nilpotent; in particular $P < N$. Hence by Lemmas 3 and 4, and the corollary of Lemma 3, one obtains

$$\varphi(|P|) \leq T_1(N, P, p') \leq T_1(G, P, p') = \varphi(|P|),$$

and $T_1(N, P, p') = T_1(G, P, p') = \varphi(|P|)$. Thus by Lemma 3(b) one has

$$N = (C(s), C(p)) \times P_2, \qquad \text{where } s|(p-1), |P:P_2| = p.$$

Hence (b) is true.

Let

$$N = LP\ (L \cap P = 1), N_1 = L_1P, L_1 \le L, |L| = s, |L_1| = s_1.$$

Then $s_1|s$. Suppose that $s_1 > 1$, i.e., $P < N_1$.

Let $\mu$ be the sum of all linear constituents of $(1_P)^G$. Obviously $\mathrm{Irr}(\mu) = \mathrm{Lin}(G/G'P)$. Since $G/G'P \cong N/N_1$ one has $\mu(1) = |N : N_1| = s/s_1$. Then we have (Sylow)

$$\big((1_P)^G - \mu\big)(1) = |G:P| - s/s_1 \equiv s - s/s_1 \not\equiv 0 \pmod p.$$

Hence there exists $\chi \in \mathrm{Irr}((1_P)^G) \cap \mathrm{Irr}_1(G, p')$.

Suppose that $P \le \ker \chi$. Let $D = P^G$ be the normal closure of $P$ in $G$. By the Frattini lemma $DN = G$, so that $G/D \cong N/N \cap D$. Now $N/N \cap D$ is abelian since $N' \le P \le N \cap D$ by (b). Hence $G' \le D$ and $G' \le \ker \chi$, a contradiction since $\chi$ is not linear. Thus $\chi \in \mathrm{Irr}_1(G, P, p')$. Since $1_P \notin \tau_1(N, P, p')_P$ (this follows from (b)), by reciprocity

$$1 + \varphi(|P|) \le T_1(G, P, p'),$$

a contradiction. Hence $s_1 = 1$, i.e., $N_1 = P$. Since $N > P$, $s > 1$ and the lemma is proved. ∎

Note that $PG'$ from Lemma 10 is $p$-nilpotent (the Burnside normal $p$-complement theorem) and solvable (Remark 1 after Proposition 9). In particular $G$ is a solvable group of $p$-length 1.

## 2

In this section we prove the first main theorem of this paper.

THEOREM A. *Suppose that $G$ is not $p$-nilpotent and*

$$T_1(G, P, p') = \varphi(|P : P'|).$$

*Then the following assertions hold*:

(a) *$G$ contains a normal subgroup $K$ such that $K \cap P = P'$, $K \leq G'$. In particular* [Is, *Theorem* 6.31] *the subgroup $K$ is $p$-nilpotent*.

(b) $\overline{G} = G/K$ *is not $p$-nilpotent*.

(c) $G/G'P \cong C(s), s > 1, s|(p-1)$.

(d) $\overline{N} = N_G(P)K/K = C(s)\overline{P} = (C(s), C(p)) \times \overline{P}_2, |\overline{P} : \overline{P}_2| = p$.

(e) $G/G' \cong C(s) \times P_2$, *where $P_2$ is a $p$-group*.

(f) *$\overline{P}$ coincides with its normalizer in $\overline{PG'}$*.

*Proof.* Let $D$ be the intersection of kernels of those characters $\lambda \in \mathrm{Lin}(P)$ for which $\mathrm{Irr}(\lambda^G) \cap \mathrm{Lin}(G)$ is non-empty. Take for each such $\lambda$ a character $\lambda^0 \in \mathrm{Irr}(\lambda^G) \cap \mathrm{Lin}(G)$. Then $\lambda^0_P = \lambda$.

Take $\mu \in \mathrm{Lin}(P) - \mathrm{Lin}(P/D)$. Then $\mathrm{Irr}(\mu^G) \cap \mathrm{Lin}(G)$ is empty. Now since $p \nmid \mu^G(1)$ there exists $\chi \in \mathrm{Irr}(\mu^G) \cap \mathrm{Irr}_1(G, p')$. By reciprocity $\chi \in \mathrm{Irr}_1(G, P, p')$. Set

$$\tau^0 = \sum \lambda^0, \qquad \textit{where } \lambda \textit{ runs over the set } \mathit{Lin}(\mathrm{P/D}).$$

Then, by reciprocity,

$$\mathrm{Lin}(P) \subseteq \mathrm{Irr}\big((\tau^0 + \tau_1(G, P, p'))_P\big).$$

By the above $T_1(G, P, p') = \varphi(|P : P'|)$ implies that $|P : D| = p$, $\langle \chi_P, \psi_P \rangle = 0$, for distinct $\chi, \psi \in \mathrm{Irr}_1(G, P, p')$, and $\chi_P$ is multiplicity free. Set $\sigma = \tau^0 + \tau_1(G, P, p')$. Since $\mathrm{Irr}(\sigma_P) = \mathrm{Lin}(P)$, $K = \ker \sigma$ satisfies $K \cap P = P'$, so $K$ is $p$-nilpotent [Is, Theorem 6.31]. Take $\nu \in \mathrm{Lin}(G)$. If $\chi \in \mathrm{Irr}_1(G, P, p')$ then $\nu\chi \in \mathrm{Irr}_1(G, p')$. Let $\chi_P = \lambda_1 + \cdots + \lambda_d$, where $\mathrm{Irr}(\chi_P) = \{\lambda_1, \ldots, \lambda_d\} \subseteq \mathrm{Lin}(P)$. Suppose that $P \leq \ker \nu\chi$. Then $(\nu\chi)_P = d \cdot 1_P$. Hence $\nu_P \lambda_i = 1_P$ for all $i$, and $\lambda_1 = \cdots = \lambda_d = \overline{\nu}_P$, a contradiction since $d = \chi(1) > 1$. Thus $\nu\chi \in \mathrm{Irr}_1(G, P, p')$ so that $\nu\tau_1(G, P, p') = \tau_1(G, P, p')$. Take $x \in G - G'$ and suppose that $\nu \in \mathrm{Lin}(G)$ is such that $\nu(x) \neq 1$. Then $\tau_1(G, P, p')(x) = 0$ and $K \leq \ker(\tau_1(G, P, p')) \leq G'$.

Let us show that $G/K$ is not $p$-nilpotent. One has $K = P'K_0$, where $K_0$ is a normal $p$-complement of $K$. Obviously $G/K_0$ is not $p$-nilpotent, so we may assume $K_0 = 1$. Then $K = P' \leq \Phi(G)$ and $G/K$ is not $p$-nilpotent by the well-known property of the Frattini subgroup [G].

Put $\overline{G} = G/K$. Then $\varphi(|\overline{P}|) = \varphi(|P : P'|)$ and

$$\varphi(|\overline{P}|) \leq T_1(\overline{G}, \overline{P}, p') \leq T_1(G, P, p') = \varphi(|P : P'|) = \varphi(|\overline{P}|);$$

so $T_1(\overline{G}, \overline{P}, p') = \varphi(|\overline{P}|)$ and $\overline{P} = PK/K \cong P/P'$ is abelian. Hence the remaining assertions of the theorem follow from Lemma 10. ∎

*Remark.* It is easy to show that the subgroup $PG'$ from Theorem A is $p$-nilpotent. Hence $PG'$ is solvable by Theorem A(f) and Remark 1 following Proposition 9. In particular, $G$ is solvable and its $p$-length is equal to 1.

## 3

We now prove the second main theorem of this paper.

THEOREM B. $T_1^0(G, p') = p - 1$ *if and only if*

$$G = (C(s), C(p)), \qquad \text{or}$$
$$G = (C(s), C(p))M = C(s)(C(p), M),$$

*semi-direct products with the kernels* $M, (C(p), M)$, *respectively.*

*Proof.* Let $M$ be a maximal normal subgroup of $G$ such that $G/M$ is not $p$-nilpotent. Lemma 5 implies

$$p - 1 = T_1^0(G, p') \geq T_1^0(G/M, p') \geq p - 1$$

so $T_1^0(G, p') = p - 1 = T_1(G/M, p')$. Therefore $M$ is defined uniquely.

(i) $M = 1$.

Then any epimorphic image of $G$ is $p$-nilpotent, so $G$ contains only one minimal normal subgroup $R$. Because $G/R$ is $p$-nilpotent, $p \mid |R|$.

(1i) $p \nmid |G:R|$.

Then $P \leq R$ and $T_1^0(G, p') = T_1(G, P, p')$. Hence, by Lemma 3(a), one has

$$p - 1 \geq T_1(G, P, p') \geq \varphi(|P:P'|), \qquad |P:P'| = p,$$
$$|P| = p, G = (C(s), C(p))$$

(the last equality follows from Theorem A). In the sequel let $p \mid |G:R|$.

(2i) $R$ is a $p$-subgroup.

Then $G = HR$, where $H$ is maximal in $G$ (since $R$ is not contained in $\Phi(G)$[G]). If $P_1 \in \mathrm{Syl}_p(H)$ is normal in $H$, then $N_G(P_1) \geq \langle H, N_R(P_1)\rangle > H$, and $P_1 > 1$ is normal in $G$, a contradiction. Obviously $C_G(R) = R$.

Let $L/R$ be a normal $p$-complement of $G/R$. Then $L$ is not $p$-nilpotent. Take $\lambda \in \mathrm{Lin}(P) - \mathrm{Lin}(P/R)$ ($R$ is not contained in $P'$ by [G]), $\chi \in \mathrm{Irr}(\lambda^G)$. As above, $R \leq G'$. Then $G'$ is not contained in $\ker \chi$ (since $R$ is not contained in $\ker \chi$), so $\chi$ is not linear, and, obviously, $G/\ker \chi = G$

is not $p$-nilpotent. Since $p \nmid \lambda^G(1)$, we may take $\chi$ so that $p \nmid \chi(1)$. Hence $\chi \in \mathrm{Irr}_1^0(G, p')$. By reciprocity

$$|\mathrm{Lin}(P) - \mathrm{Lin}(P/R)| = |P : P'| - |P : P'R| \leq T_1^0(G, p').$$

Since $P$ is not cyclic, $|P| \geq p^2$. Now $RP' < P$ and $|P : P'| - |P : P'R| \geq \varphi(|P : P'|) > p - 1$, a contradiction.

(3i) $R$ is not $p$-solvable.

Then $R$ is not $p$-nilpotent so $P_1 = P \cap R$ is not contained in $P'$ [Is, Theorem 6.31]. Take

$$\lambda \in \mathrm{Lin}(P) - \mathrm{Lin}(P/P_1), \qquad \chi \in \mathrm{Irr}(\lambda^G) \cap \mathrm{Irr}_1(G, p').$$

Then $R < G'$, $\chi \in \mathrm{Irr}_1^0(G, p')$, and by reciprocity

$$T_1^0(G, p') \geq |\mathrm{Lin}(P) - \mathrm{Lin}(P/P_1)| = |P : P'| - |P : P'P_1| > p - 1,$$

a contradiction.

(ii) $M > 1$.

Then $G/M \cong (C(s), C(p))$ by (i). As in the proof of Theorem A, we may show that $M = \ker(\tau_1^0(G, p')) < G'$; the equality $M = \ker \tau_1^0(G, p')$ follows from

$$T_1^0(G/M, p') = p - 1 = T_1^0(G, p').$$

Thus $|G' : M| = p, G/G' \cong C(s)$. It remains to prove that $G' = (C(p), M)$.

Take $\chi \in \mathrm{Irr}(G, P, p')$. Since $p \,|\, |G/\ker \chi|$ and $p \nmid |G : G'|, G/\ker \chi$ is not $p$-nilpotent, i.e., $\chi \in \mathrm{Irr}_1^0(G, p')$. Therefore

$$\mathrm{Irr}_1(G, P, p') \subseteq \mathrm{Irr}_1^0(G, p').$$

Since the opposite inclusion is evident, one has

$$T_1(G, P, p') = T_1^0(G, p') = p - 1.$$

Then Theorem A implies $|P| = p$ and $N_{G'}(P) = P$ so $G' = (C(p), M)$. Since a group $G$ with such a structure satisfies the condition of the theorem, the proof is finished. ∎

In the same way, using Lemma 8, we may prove the following:

Proposition 11. *Suppose that $G$ is not $p$-nilpotent and $P \in \mathrm{Syl}_p(G)$ is cyclic. Then the following conditions are equivalent:*

(a) $T_1^0(G, p') = |P| - 1$.

(b) $G = (C(s), P)M$, *a semi-direct product with normal subgroup* $M$, $N_{PM}(P) = P$.

Obviously $G$ from Proposition 11 is solvable.

## 4

Let

$$\mathrm{Irr}_1^1(G, p') = \{\chi \in \mathrm{Irr}_1(G, p') | G/\ker\chi \text{ is not } p\text{-solvable}\},$$

$$\tau_1^1(G, p') = \sum \chi, \qquad \text{where } \chi \text{ runs over the set } \mathrm{Irr}_1^1(G, p');$$

$$T_1^1(G, p') = \tau_1^1(G, p')(1).$$

In this section we give a lower estimate of $T_1^1(G, p')$ for a non-$p$-solvable $G$ with a cyclic Sylow $p$-subgroup $P$.

Lemma 12. *Let a non-$p$-nilpotent group $G = AP$, where $P \cong C(p^m)$, $P \in \mathrm{Syl}_p(G)$ is normal in $G$, $A \cap P = 1$. If $T_1(G, P, p') \le |P| + 2$ then one of the following assertions holds*:

(a) $p = 3, G \cong D(12)$, *a dihedral group of order* 12.

(b) $p = 3, G = C(4)C(3)$.

(c) $G = (C(s), P), s|(p - 1)$.

*Proof.* Set $L = C_A(P)$. Then $G/L \cong (C(s), P), s|(p-1)$. If $L = 1$ one obtains a group (c). Assume that $L > 1$. Obviously $p \ne 2$. We have $T_1(G/L, PL/L, p') = |P| - 1$ (see [Is, Theorem 6.34]). Let $\mu \in \mathrm{Irr}^\#(L)$, $\lambda \in \mathrm{Lin}(P)$ with $\ker\lambda = 1$, and $\chi \in \mathrm{Irr}((\mu \times \lambda)^G)$. Since $P \le G'$ (Lemma 8) and $\ker\chi \cap P = 1, \chi(1) > 1$. Since $\mathrm{Irr}_1(G) = \mathrm{Irr}_1(G, p'), \chi \in \mathrm{Irr}_1(G, P, p')$. Hence

$$T_1(G, P, p') \ge T_1(G/L, PL/L, p') + \chi(1) = |P| - 1 + \chi(1).$$

By condition $\chi(1) \le 3$. If there is a character $\tau \in \mathrm{Irr}_1(G, P, p') - \mathrm{Irr}(G/L, PL/L, p') - \{\chi\}$ we obtain

$$T_1(G, P, p') \ge |P| - 1 + \chi(1) + \tau(1) > |P| + 2,$$

a contradiction. Thus

$$\mathrm{Irr}_1(G, P, p') = \mathrm{Irr}_1(G/L, PL/L, p') \cup \{\chi\}, \chi(1) \le 3.$$

Then all characters from $\mathrm{Irr}^\#(L)$ are $G$-conjugate. This implies that $L$ is an elementary abelian $q$-group, a prime $q \ne p$. Moreover $L$ is a minimal normal subgroup of $G$. Suppose that $|P| > p$, $P_1$ is a subgroup of order $p$ in $P$. Take in $\mathrm{Lin}(P/P_1)$ a faithful character $\lambda_1$, and take $\mu \in \mathrm{Irr}^\#(L)$. If $\tau \in \mathrm{Irr}((\mu \times \lambda_1)^G)$ then $\tau \ne \chi$ and $\tau \in \mathrm{Irr}_1(G, P, p')$, a contradiction. Thus $|P| = p$. Suppose that $L$ is not contained in $G'$; then $L = Z(G)$ and

$|L| = 2$ (since all characters from $\mathrm{Lin}^{\#}(L)$ are $G$-conjugate). Hence $G = D(4p)$, a dihedral group of order $4p$, or $G = C(4)P$. Then $T_1(G, P, p') = 2(p-1) \leq p+2$, so $p = 3$ and $G \cong D(12)$ or $G = C(4)C(3)$.

Suppose that $L \leq G'$. Then $G' = P \times L$ and

$$|G| - |G|/|P| - |G|/|L| + |G|/|P||L| = \chi(1)^2 \in \{4, 9\}.$$

Hence $s|L|p - sp - s|L| + s = t \in \{4, 9\}$.

(i) Suppose that $t = 4$. Then $s|4$. If $s = 4$ then $|L|p = |L| + p$ and $p = 2$, a contradiction. Let $s = 2$. Then $|L|p = |L| + p + 1$, a contradiction.

(ii) Now let $t = 9$. Then $s|9$. If $t = 9$ then $|L|p = p + |L|$ and $p = 2$, a contradiction. If $s = 3$ then $|L|p = p + |L| + 2$, which is impossible. The lemma is proved. ▌

THEOREM 13. *Let $G$ be a non-$p$-solvable group with cyclic Sylow $p$-subgroup $P$. Then*

(a) $T_1^1(G, p') \geq |P| + 1$.

(b) *If $p > 3$ then* $T_1^1(G, p') \geq |P| + 2$.

*Proof.* Let $G$ be a counterexample of minimal order. Then

(i) $O_{p'}(G) = 1$.

Let $R > 1$ be a normal subgroup of $G$ such that $P$ is not contained in $R$. Then $R \cap P \leq \Phi(P)$, and hence $R$ has a normal $p$-complement $O_{p'}(R)$ ($\leq O_{p'}(G) = 1$) ([Is, Theorem 6.31]). Thus $R \leq P$. By the transfer theorem $P \cap C_G(R)' \cap Z(C_G(R)) = 1$ or $R \cap C_G(R)' = 1$. Therefore $C_G(R)' \leq O_{p'}(C_G(R))$. Since $C_G(R)$ is normal in $G$,

$$O_{p'}(C_G(R)) \leq O_{p'}(G) = 1 \qquad \text{and} \qquad C_G(R) = P.$$

Since $G/C_G(R)$ is isomorphic to a subgroup of $C(p-1)$, $G$ is solvable, a contradiction. Thus

(ii) Any non-trivial normal subgroup of $G$ contains $P$ and, in particular, it is not $p$-solvable. Moreover $G$ contains only one minimal normal subgroup $R$. Since $P < R$, $G/R$ is a $p'$-group. Now $R \leq G'$ by Lemma 8.

(i)–(ii) imply

(iii) $\mathrm{Irr}_1(G, P, p') = \mathrm{Irr}_1^1(G, p')$, $T_1(G, P, p') = T_1^1(G, p')$.

Set $N = N_G(P)$, $N_1 = N \cap R$ $(= N_R(P))$.

By Burnside's normal $p$-complement theorem one has

(iv) $N$ and $N_1$ are not $p$-nilpotent.

(v) $T_1(G, P, p') \geq T_1(N, P, p')$.

For the proof see Lemma 4(a).

Now suppose that $R < G$.

By the assumption on $G$ and (v) it follows that $T_1(N, P, p') \leq |P| + 2$. By virtue of Lemma 12 we know the structure of $N$. By the Frattini lemma one has $G = RN$ so that $R = G'$. Now Lemma 12 and (v) imply

(vi) If $N$ is not a Frobenius group then $N \cong D(12)$, $N_1 \cong D(6)$.

Suppose that $N \cong D(12)$, $N_1 \cong D(6)$. Then $|P| = 3$ and $C_R(P) = P$. So by the Feit–Thompson theorem one has $R \in \{A_5, PSL(2,7)\}$ [FT]. Hence, by (i), one has $G \in \{S_5, PGL(2,7)\}$. Since

$$T_1(S_5, P, 3') = 18 > 5 = |P| + 2,$$
$$T_1(PGL(2,7), P, 3') = 30 > 5 = |P| + 2,$$

we obtain a contradiction. Thus

(vii) If $R < G$ then $N = (C(s), P)$, $N_1 = (C(s_1), P)$, $s_1 | s$, $s_1 < s$.

We continue to consider the case $R < G$.

Take $\lambda \in \mathrm{Lin}(N) - \mathrm{Lin}(N/N_1)$. Since $\lambda^G(1) = |G : N| \equiv 1 \pmod p$ (Sylow) and $G' = R$ is not contained in $\ker \lambda$ there exists (by reciprocity) a $\chi \in \mathrm{Irr}(\lambda^G) \cap \mathrm{Irr}_1(G, p')$. Now

$$|\mathrm{Lin}(N)| - |\mathrm{Lin}(N/N_1)| = s - s/s_1.$$

Thus

$$T_1(G, P, p') \geq T_1(N, P, p') + s - s/s_1 = |P| - 1 + s - s/s_1.$$

In particular $T_1(G, P, p') \geq |P| + 1$, and (a) is proved in the case $R < G$. Next we assume that $p > 3$, $s - s/s_1 = 2$. Then

$$s = 4, s_1 = 2, N = (C(4), P), 4 | (p - 1).$$

Let $\varphi \in \mathrm{Irr}_1(N)$, $\chi \in \mathrm{Irr}(\varphi^G) \cap \mathrm{Irr}_1(G, p')$. Then $\chi \in \mathrm{Irr}_1(G, P, p')$. If for some choice of $\varphi$ one has $|\mathrm{Irr}(\varphi^G) \cap \mathrm{Irr}_1(G, p')| > 1$ then $\varphi(1) = 4$ implies (by reciprocity)

$$T_1(G, P, p') \geq T_1(N, P, p') + \varphi(1) \geq |P| - 1 + 4 = |P| + 3,$$

and the theorem is proved. Thus we may assume that for any $\varphi \in \mathrm{Irr}_1(N)$ one has $|\mathrm{Irr}(\varphi^G) \cap \mathrm{Irr}_1(G, p')| = 1$. Now for $\chi$ from $\mathrm{Irr}(\varphi^G) \cap \mathrm{Irr}_1(G, p')$ one has $\langle \varphi^G, \chi \rangle = 1$ (since in the contrary case the contribution of $\varphi$ in $T_1(G, P, p')$ is at least 8, and we obtain $T_1(G, P, n') \geq |P| - 1 + 4 = |P| + 3$). In particular for our $\chi$ one has $\chi(1) \equiv \varphi^G(1) \equiv 4 \pmod p$.

Take $\tau \in \mathrm{Irr}_1(G, P, p') = \mathrm{Irr}_1(G, p')$. Then $\tau$ is faithful. Hence $\tau_N$ has a non-linear irreducible constituent $\psi$. By the above, in view of $\tau \in \mathrm{Irr}(\psi^G) \cap \mathrm{Irr}_1(G, P, p')$, it follows that $\tau(1) \equiv 4 \pmod p$.

Take $\lambda \in \mathrm{Lin}(N) - \mathrm{Lin}(N/N_1)$. Then $\lambda^G(1) = |G:N| \equiv 1 \pmod p$ and $\mathrm{Irr}(\lambda^G) \subseteq \mathrm{Irr}_1(G)$ (since any irreducible constituent of $\lambda^G$ does not contain $G'$ in its kernel). Hence there exists $\chi \in \mathrm{Irr}(\lambda^G) \cap \mathrm{Irr}_1(G, p')$. If the contribution of $\lambda$ in $T_1(G, p')$ is at least 2 then $T_1(G, p') \geq |P| - 1 + s - s/s_1 + 1 \geq |P| + 2$. Hence we may assume that $\langle \lambda^G, \chi \rangle = 1$ for any $\chi \in \mathrm{Irr}(\lambda^G) \cap \mathrm{Irr}_1(G, p')$. By the proved $\chi(1) \equiv 4 \pmod p$. Since $p > 3$ there exists $\tau \in \mathrm{Irr}((\lambda)^G - \chi) \cap \mathrm{Irr}_1(G, p')$ (because $p \nmid (\lambda^G) - x)(1))$ so that the contribution of $\lambda$ in $T_1(G, p')$ is greater than 1, a contradiction. Thus

(viii) $G' = R = G$; i.e., $s = s_1$, $N = N_1$, $G$ is a non-abelian simple group.

Suppose that $N$ is not a Frobenius group. By Lemma 12 one has $N = D(12)$ or $N = C(4)C(3), |P| = 3$. Then $T_1(N, 3') = 4 = |P| + 1$, and (a) is proved by virtue of (v). Therefore in the sequel we assume that $p > 3$. Then $N = (C(s), P)$ by (v) and Lemma 12.

As above it is easy to prove the following assertions:

(ix) If $\varphi \in \mathrm{Irr}_1(N)$ then $|\mathrm{Irr}(\varphi^G) \cap \mathrm{Irr}_1(G, p')| = 1$. Moreover if $\chi \in \mathrm{Irr}(\varphi^G) \cap \mathrm{Irr}_1(G, p')$ then $\langle \varphi^G, \chi \rangle = 1$.

(x) $\chi(1) \equiv s \pmod p$ for all $\chi \in \mathrm{Irr}_1(, P, p') = \mathrm{Irr}_1(G, p')$.

(xi) If $\lambda \in \mathrm{Lin}^{\#}(N)$ then by virtue of (x) one has $|\mathrm{Irr}(\lambda^G) \cap \mathrm{Irr}_1(G, p')| > 1$ since $\lambda^G(1) \equiv 1 \pmod p$, $\mathrm{Irr}(\lambda^G) \subseteq \mathrm{Irr}_1(G)$. Hence

(xii) $T_1(G, P, p') \geq T_1(N, p') + 2(s - 1) = |P| - 1 + 2(s - 1)$.

Since $G$ is a counterexample,

(xiii) $s = 2$.

If, in (xi), we have $|\mathrm{Irr}(\lambda^G) \cap \mathrm{Irr}_1(G, p')| > 2$ then we obtain in (xii) strong inequality, so $T_1(G, P, p') > |P| - 1 + 2 = |P| + 1$, a contradiction (since $G$ is a counterexample). Now $\chi(1) \equiv 2 \pmod p$ for all $\chi \in \mathrm{Irr}_1(G, p')$ by (x).

Let $\lambda \in \mathrm{Lin}^{\#}(N)$. Because $\mathrm{Irr}(\lambda^G) \subseteq \mathrm{Irr}_1(G), \lambda^G(1) \equiv 1 \pmod p$, (x) implies that $|\mathrm{Irr}(\lambda^G) \cap \mathrm{Irr}_1(G, p')| \geq (p + 1)/2 > 2$, a contradiction with the above. Then theorem is proved. ∎

*Conjectures.* 1. Suppose that $G$ and $P$ are as in Theorem 13. Then

$$T_1^1(G, p') \geq |P| + p.$$

2. If $G$ is a non-$p$-solvable group then $T_1^1(G, p') \geq 2p$.

QUESTION. *Suppose that $P \in Syl_p(G)$ and $P < H < G$. Classify all pairs $H < G$ such that $T_1(H, P, p') = T_1(G, P, p')$.*

## 5

In this section we consider groups $G$ with $|\mathrm{Irr}_1(G, p')| = 1$.

PROPOSITION 14. *Suppose that $p$ is a prime divisor of $|G|$, $|\mathrm{Irr}_1(G, p')| = 1$, $\mathrm{Irr}_1(G, p') = \{\chi\}$, $\ker\chi = 1$. Then the following assertions hold*:

(a) *$G'$ is the only minimal normal subgroup of $G$.*

(b) $p \nmid |G:G'|$.

(c) *If $G$ is $p$-solvable then $G = (C(p^n - 1), E(p^n))$.*

(d) *If $p = 2$ then $G' < G$.*

(e) *If $G$ is not $p$-solvable, then $G'$ is simple and all characters from $\mathrm{Irr}_1(G', p')$ are conjugate under $G$.*

*Proof.* Suppose that $p \mid |G:G'|$. Then $G$ contains a normal subgroup $H$ of index $p$. By Clifford's theory $\chi_H = \eta \in \mathrm{Irr}(H)$, and

$$\eta^G = \chi_1 + \cdots + \chi_p,$$

where $\chi_1, \ldots, \chi_p$ are distinct irreducible characters of $G$ of $p'$-degree $\chi(1)$, a contradiction. Hence (b) is proved.

Let $N$ be a minimal normal subgroup of $G$. By the condition the degrees of all non-linear irreducible characters of $G/N$ are divisible by $p$, so $G/N$ has a normal $p$-complement $H/N$ (Thompson) and (b) implies $H = G$ whence $G/N$ is a $p'$-number. Since $N > \ker\chi = 1$ all irreducible characters of $G/N$ are linear by the Ito theorem [Is], Theorem 6.15]. Hence $G/N$ is abelian, $N = G'$, and (a) is proved.

By (b) one has $P \leq G'$ (here $P \in \mathrm{Syl}_p(G)$). Suppose $G$ is $p$-solvable. Then $P = G'$ is a minimal normal subgroup of $G$ by (a). Hence $\mathrm{Irr}_1(G) = \mathrm{Irr}_1(G, p')$ ([Is, Theorem 6.15]) and $|\mathrm{Irr}_1(G)| = 1$. Therefore $G = (C(p^m - 1), E(p^m))$ by [S] and (c) is proved.

Suppose that $p = 2$ and $G' = G$. Since all characters from $\mathrm{Irr}(G) - \{1_G, \chi\}$ have even degrees, $|G| \equiv 1 + \chi(1)^2 \equiv 2 \pmod 4$. Therefore $G$ contains a normal 2-complement, contradicting (b).

Suppose that $G$ is not $p$-solvable. Let $\chi_{G'} = e(\lambda_1 + \cdots + \lambda_t)$ be the Clifford decomposition. Since $G'$ is a direct product of isomorphic non-abelian simple groups, $\lambda_1, \ldots, \lambda_t$ are non-linear. Suppose that $\lambda \in \mathrm{Irr}_1(G', p') - \{\lambda_1, \ldots, \lambda_t\}$. Then by (b) and the Clifford theory one has $\mathrm{Irr}(\lambda^G) \subseteq \mathrm{Irr}_1(G, p')$. Since $\chi \notin \mathrm{Irr}(\lambda^G)$ one obtains a contradiction. Thus we have $\mathrm{Irr}_1(G', p') = \{\lambda_1, \ldots, \lambda_t\}$, and $\lambda_1, \ldots, \lambda_t$ are conjugate under $G$. In particular $\lambda_1(1) = \ldots = \lambda_t(1)$. Hence $G'$ is not a direct product of two or more non-abelian simple groups. This proves (e). The proposition is proved. ∎

*Remark* 1. If $\mathrm{Irr}_1(G, p') = \{\chi\}$ then $\ker\chi$ is $p$-nilpotent and solvable (see Remark 1 after Proposition 9).

*Remark* 2. Put $c(G, p') = |\mathrm{Irr}_1(G, p') \cup \mathrm{Lin}(G)|$. Alperin and McKay conjectured that $c(G, p') = c(N_G(P), p')$ (see [Mic], p. 227). If a non-$p$-solvable group $G$ satisfies $|\mathrm{Irr}_1(G, p')| = 1$ and $G/\ker\chi$ satisfies the Alperin–McKay conjecture (we do not assume here that this conjecture holds in general) then it is easy to prove the following assertions:

(a) $p < 5$.

(b) $N_G(P)/P' = (C(p^d - 1), E(p^d))$, where $|P : P'| = p^d$. Moreover $N_{G'}(P) = P$, $P' = \Phi(P)$.

(c) $G/G' \cong C(p^d)$.

*Conjecture.* If $|\mathrm{Irr}_1(G, p')| = 1$ then $G$ is $p$-solvable.

## 6

In this section we prove the following

PROPOSITION 15. *Let $P \in \mathrm{Syl}_p(G)$. Then the following assertions are equivalent*:

(a) $T_1(G, p') = 0$.

(b) $T_1(G, P, p') = 0$.

*Proof.* Obviously $T_1(G, p') = 0$ implies $T_1(G, P, p') = 0$.

Assume that (b) is true but (a) is not true. Let $P^G = L$ be the normal closure of $P$ in $G$ and $H$ the normal $p$-complement of $G$ (note that $G$ is $p$-nilpotent by Lemma 7). Then $N = N_G(P) = P \times A$. Suppose that $A$ is not abelian. Take $\mu \in \mathrm{Irr}_1(A)$ and $\lambda \in \mathrm{Lin}^{\#}(P)$. Since $p \nmid (\lambda \times \mu)^G(1)$ there exists $\chi \in \mathrm{Irr}((\lambda \times \mu)^G)$ such that $p \nmid \chi(1)$. Now $\chi \in \mathrm{Irr}_1(G, P, p')$ by reciprocity, a contradiction. Thus $A$ is abelian. By the Frattini lemma one has $G = LN = LA$. Suppose that $\tau \in \mathrm{Irr}_1(G, p')$. By the condition one has $P \le \ker\tau$. Hence $P^G = L \le \ker\tau$, a contradiction, since $G/L$ is abelian and $\tau$ is non-linear. Thus (a) is true and the proposition is proved. ∎

Let $G = P \times A$, where $P \in \mathrm{Syl}_p(G)$ and $A$ is not abelian. Then $T_1(G, p') > T_1(G, P, p')$.

PROPOSITION 16. *Let $P \in \mathrm{Syl}_p(G)$, $L = P^G$. Then the following assertions are equivalent*:

(a) $T_1(G, p') = 0$.

(b) $T_1(L, p') = 0$ *and* $G' = L'$.

*Proof.* Suppose that (a) is true. Then $T_1(L, p') = 0$ by Lemma 4(b). To prove $G' = L'$ we may assume that $L' = 1$. Then $G$ is abelian by Proposition 9 and $G' = 1$. Thus (b) is proved. Suppose that (b) is true but (a) is not true. Take $\chi \in \mathrm{Irr}_1(G, p')$. By (b) and the Clifford theorem one has $\mathrm{Irr}(\chi_L) \subseteq \mathrm{Lin}(L)$. Hence by (b) we have $G' = L' \leq \ker \chi$ and $\chi$ is linear, a contradiction. ▌

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